MODEL TEST PAPER - III

Time: 3 hours **Maximum Marks: 100**

General Instructions:

1. Find the argument of complex number z = $\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}$

   Solution. Z = $\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}$

   $\Rightarrow$ z = $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

   So, arg(z) = $\frac{\pi}{3}$.

2. Evaluate : $\lim_{x\to 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$

   Solution. $\lim_{x\to 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$ let $\frac{1}{x} = y$

   = $\lim_{y\to\infty}\frac{\sin y}{y} = 0$

3. Find the number of terms in the expansion of $(3x+y)^8 - (3x-y)^8$

   Solution. 4 terms.

4. Write the domain of the function, $f(x) = \frac{x}{x^2 - 5x + 6}$

Solution. f(x) $\frac{x}{x^2-5x+6}=\frac{x}{(x-3)(x-2)}$

For Domain (f) = R–{3, 2}

5. Two finite set have m and n element. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m and n.

Solution. Let A and B are two sets having m and n elements. A.T.Q

$2^m - 2^n = 56$

$\Rightarrow 2^n(2^{m-n} - 1) = 8 \times 7$

$\Rightarrow 2^n(2^{m-n} - 1) = 2^3 \times (2^3 - 1)$

As comparing, n = 3; m – n = 3

$\Rightarrow$ m = 6

Thus, m = 6; n = 3.

6. Let f : R $\rightarrow$ R be a function given by $f(x) = x^2 + 1$. Find $f^{-1}(-5)$.

Solution. let $f^{-1}(-5) = x \Rightarrow f(x) = -5$

$\Rightarrow x^2 + 1 = -5$

$\Rightarrow x^2 = -6$

$\Rightarrow$ x = no real value.

So, $f^{-1}(-5) = \phi$

7. If $\frac{a+ib}{c+id}=x+iy$ prove that $\frac{a+ib}{c+id}=x-iy$.

Solution $\because$ $\frac{a+ib}{c+id}=x+iy$ [Given]

$$\Rightarrow \overline{\left(\frac{a+ib}{c+id}\right)} = \overline{x+iy} \quad [\text{If } z_1 = z_2 \Rightarrow \overline{z}_1 = \overline{z}_2]$$

$$\Rightarrow \frac{(\overline{a+ib})}{(\overline{c+id})} = x - iy \quad \left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}\right]$$

$$\Rightarrow \frac{a-ib}{c-id} = x - iy$$

8. If (n +1) ! = 12 (n – 1) !, find n.

Solution. (n + 1)! = 12 (n – 1) !

$\Rightarrow$ (n + 1) . n . (n – 1)! = 12 (n – 1) !

$\Rightarrow$ (n + 1) n = 12

$\Rightarrow$ (n + 1) n = 4 × 3

$\Rightarrow$ n = 3

9. Find the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$

Solution. In the expansion of $\left(\frac{x}{3}+9y\right)^{10}$, the middle term is T_6.

$T_6 = 10_{C_5}\left(\frac{x}{3}\right)^5 (9y)^5$

$= \frac{\lfloor 10}{\lfloor 5 \lfloor 5}\frac{x^5}{3^5} \times 9^5 y^5$

$= 252 \times 3^5 x^5 y^5$

$= 61236\, x^5 y^5$.

10. Find the sum of first 24 te4rms of the A. P.

$a_1, a_2, a_3.$ .... if it is known that

$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225.$

Solution. $\because a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{20}) = 225$

$\Rightarrow 3(a_1 + a_{24}) = 225$ $\left[\begin{array}{l}\because a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...\\ \text{in an A.P}\end{array}\right]$

$\Rightarrow a_1 + a_{24} = 75$

Now, $S_{24} = \frac{24}{2}(a_1 + a_{24})$

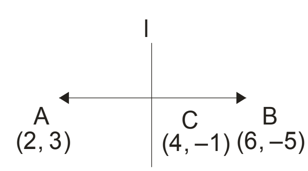
$= 12 \times 75 = 900.$

11. Find the equation of the perpendicular bisector of the line segment joining the points A(2, 3) and B(6, –5).

Solution. Slope of AB = $\frac{-5-3}{6-2} = \frac{-8}{4} = -2$

$\because l \perp AB,$

So, slope of line l is m = $\frac{1}{2}$



equation of line l is

$y + 1 = \frac{1}{2}(x-4)$

$\Rightarrow$ x – 2y – 6 = 0.

12. Find the derivative of sin x. Cos x w.r.t. 'x'

Solution. y = sin x cos x

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

= sin x(– sin x) + cos x . cos x

= – $\sin^2$ x + $\cos^2$ x

= cos 2x.

13. Show that $\sqrt{3}\ \text{cosec}\ 20° - \sec 20° = 4$

Solution. LHS= $\sqrt{3}\ \text{cosec}\ 20° - \sec 20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cos 20°}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{\sin 20° \cos 20°} = \frac{2(\sin 60° \cos 20° - \cos 60° \sin 20°)}{\sin 20° \cos 20°}$$

$$= \frac{2\sin(60 - 20°)}{\sin 20° \cos 20°} = \frac{2\sin 40°}{\sin 20 \cos 20°} = \frac{4\sin 40}{2\sin 20° \cos 20°}$$

$$= \frac{4\sin 40°}{\sin 40°} = 4$$

14. Solve (x + iy) (2 – 3i) = 4 + i, where x and y are seal

Solution. (x + iy) (2 – 3i) = 4 + i

$$\Rightarrow x + iy = \frac{4+i}{2-3i} = \frac{(4+i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)}$$

= $\frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i$

So, x = $\frac{5}{13}$ and $y = \frac{14}{13}$.

15. Let P be the solution set of 3x + 1 > x – 3 and is Q be the solution set of 5x + 2 ≤ 3(x + 2), x ∈ n. Find the set P ∩ Q

Solution ∵ 3x + 1 > x – 3

⇒ 3x – x > –3 –1

⇒ 2x > – 4

⇒ x > –2

But x ∈ N. ∴, P = {1, 2, 3 ...}

Also, 5x + 2 ≤ 3(x + 2)

⇒ 5x – 3x ≤ 6 – 2

⇒ 2x ≤ 4

⇒ x ≤ 2

But x ∈ N, ∴ Q = {1, 2}

∴ P ∩ Q = {1, 2}

16. If there are six periods in each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least on period?

Solution. Six periods can be arranged for 5 subject in 6/5 ways.

= 720 ways.

One periods is left, which can be arranged for any of the five subject, one left period can be arranged in 5 ways.

Required no, of arrangements = 720 × 5 = 3600.

17. Find the term in dependent of x in $2x^2 \quad \frac{1}{3x^3} \quad {}^{10}$.

Solution. General term, T_{r+1} = $10_{C_r}(2x^2)^{10\ r} \quad \frac{1}{3x^3} \quad {}^{r}$

$= 10_{C_r} 2^{10 \quad r} \quad \frac{1}{3}^{\,r} x^{20 \quad 5r}$

It will be independent of x if 20 – 5r = 0, i.e. if r = 4

so, T_5 = $10_{C_4} \quad 2^6 \quad \frac{1}{2}^{4} \quad \frac{4480}{27}$.

18. Divide 63 into three parts such that they are in G.P. and the product of the first and the second term is $\frac{3}{4}$ of the third term.

Solution. Let the three numbers be a, ar, ar^2.

Given a + ar + ar^2 = 63 ...(1) and a. ar = $\frac{3}{4}ar^2$

$\Rightarrow$ a = $\frac{3}{4}r$ ...(2)

From (1) and (2) are get

$$\frac{3}{4}r + \frac{3}{4}r^2 + \frac{3}{4}r^3 = 63$$

$\Rightarrow$ $r^3 + r^2 + r - 84 = 0$

$\Rightarrow$ $(r - 4)(r^2 + 5r + 21) = 0$

$\Rightarrow$ r = 4, $\frac{-5 \pm \sqrt{25 - 84}}{2}$

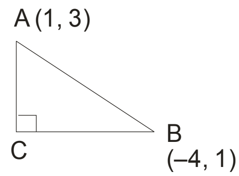
Real value of r is 4. So, a = 3.

$\therefore$, Three numbers are 3, 12, 48,

19. The hypotenuse of a right angled triangle has its ends at the points ((1, 3)) and (–4, 1). Find the equation of the legs of the triangle.

Solution. Let ABC be the right angled triangle such that $\angle c = 90°$

Let m be the slope of the line AC then the slope of BC = $\frac{1}{m}$.



Equation of AC is : y – 3 = m(x – 1) and equation of BC is

$y = -1 - \frac{1}{m}(x+4).$

or $x - 1 = \frac{1}{m}(y-3)$

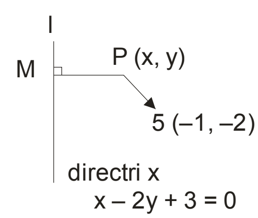
For m = 0, these lines are x + 4 = 0, y – 3 = 0

For m = ∞, the lines are x – 1 = 0, y – 1 = 0.

20. Find the equation of parabola whose focus at (–1, –2) and directrix is x – 2y + 3 = 0

Solution. Let P(x, y) be any point on the parabola is using focus-directrix property of the parabola, SP = PM

$\therefore \sqrt{(x \quad 1)^2 \quad (y \quad 2)^2} = \frac{|x-2y+3|}{\sqrt{1^2+(-2)^2}}$



$\Rightarrow (x + 1)^2 + (y + 2)^2 = \frac{(x-2y+3)^2}{5}$

$\Rightarrow 5x^2 + 5 + 10x + 5y^2 + 20 + 20y = x^2 + 4y^2 + 9 - 4xy - 12y + 6x$

$\Rightarrow 4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$. This is required equation of parabola

21. Evaluate : $\lim_{x \to \sqrt{3}} \frac{x^2 - 3}{x^2 + 3\sqrt{3}x - 12}$

Solution. $\lim_{x \to \sqrt{3}} \frac{x^2 - 3}{x^2 + 3\sqrt{3}x - 12}$ = $\lim_{x \quad \sqrt{3}} \frac{(x \quad \sqrt{3})(x \quad \sqrt{3})}{x^2 \quad 4\sqrt{3}x \quad \sqrt{3}x \quad 12}$

= $\lim_{x \to \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{(x + 4\sqrt{3})(x - \sqrt{3})} = \lim_{x \to \sqrt{3}} \frac{(x + \sqrt{3})}{(x + 4\sqrt{3})}$

= $\frac{2\sqrt{2}}{5\sqrt{3}} = \frac{2}{5}$

22. In a single throw of three dice, determine the probability of getting total of at most 5.

Solution. Number of exhaustive cases in a single throw of three dice = 6 × 6 × 6 = 216. (favarouble number of cases = 10{i.e. (1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1)}

So, required Probolility = $\frac{10}{216} = \frac{5}{108}$.

23. Let f be defined by f(x) = x – 4 and g be defined by

g(x) = $\begin{cases} \frac{x^2 - 16}{x + 4}, & x \neq -4 \\ k, & x = -4 \end{cases}$

Find k such that f(x) = g(x) for all x.

Solution. we have f(–4) = –4 –4 = –8 and g(–4) = k.

But f(x) = g(x) $\forall$ x.

$\therefore$, –8 = k i.e. k = –8 Ans.

24. Calculate the mean deviation from the median of following data.

Wages per week (in Rs)	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of workers	4	6	10	20	10	6	4

Solution.

Wages per Week in Rs	Mid value x_i	Frequency f_i	Cumulative frequency	Deviation $\lvert d \rvert = \lvert x_i - 45 \rvert$	$f_i \lvert d_i \rvert$
10-20	15	4	4	30	120
20-30	25	6	10	20	120
30-40	35	10	20	10	100
40-50	45	20	40	0	0
50-60	55	10	50	10	100
60-70	65	6	56	20	120
70-80	75	4	60	30	120
		$N = f_i = 60$			$\Sigma f_i \lvert di \rvert = 680$

Here N = 60, so, $\frac{N}{2}$ = 30; Median = $l + \left(\frac{\frac{n}{2} - f_c}{f_m} \right) \times h$

= $40 + \left(\frac{30 - 20}{20} \right) \times 10 = 45$

Mean definition from median = $\frac{f_i \lvert di \rvert}{N} \quad \frac{680}{60} \quad 11.33$ Ans.

25. If p and p' be the perpendiculars from the orign upon the straight lines x sec θ – y cosec θ = a and x cos θ + y sin θ = a cos 2θ prove that $4p^2 + p'^2 = a^2$.

Solution. one line is x sec θ – y cosec θ – a = 0 ...(1)

P = length of perpendicular from the origin (0, 0) on (1)

$$= \left|\frac{-a}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}\right| = \left|\frac{-a}{\sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}}\right| = \left|\frac{-a}{\frac{1}{\sin\theta\cos\theta}}\right|$$

$\Rightarrow$ p = a sin θ cos θ ...(2)

The other line is x cos θ + y sin θ – a cos 2θ = 0 ...(3)

P' = length of perpendicular from origin (0, 0) on (3) is

$$= \left|\frac{a\cos 2}{\sqrt{\cos^2 \quad \sin^2}}\right| \quad a\cos 2$$

$\therefore$, $4p^2 + p'^2 = 4a^2 \cos^2\theta \sin^2\theta + a^2 \cos^2 2\theta$

$= a^2(2\cos\theta \sin\theta)^2 + a^2 \cos^2 2\theta$

$= a^2 \sin^2 2\theta + a^2 \cos^2 2\theta$

$= a^2(\sin^2 2\theta + \cos^2 2\theta)$

$= a^2$

Hence $4p^2 + p'^2 = a^2$.

26. Sum the series $\frac{1^3}{1} + \frac{1^3+2^3}{2} + \frac{1^3+2^3+3^3}{3} + ...$ to n terms.

Solution. Here

$$t_n = \frac{1^3+2^3+3^3+...+n^3}{n} = \frac{\sum_{k=1}^{n} k^3}{n} = \frac{n^2(n+1)^2}{4n}$$

$$= \frac{n}{4}(n^2+2n+1) = \frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n$$

$$S_n = \frac{1}{4}\sum_{k=1}^{n} k^3 + \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{4}\sum_{k=1}^{n} k$$

$$= \frac{1}{4}\cdot\frac{n^2(n+1)^2}{4} + \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{4}\cdot\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{48}[3n(n+1)+4(2n+1)+6]$$

$$= \frac{n(n+1)}{48}(3n^2+1(1n+10) = \frac{n(n+1)(n+2)(3n+5)}{48}$$

27. For any two sets A and B, prove that P(A) = P(B) $\Rightarrow$ A = B

Solution. Let x be an arbitrary element of A. Then, there exists a subset, say X, of set A such that x $\in$ X. Now,

$X \subset A \Rightarrow X \in P(A)$

$\Rightarrow X \in P(B)$ [$\because$ P(A) = P(B)]

$\Rightarrow X \subset (B)$

$\Rightarrow x \in B$ [$\because x \in X$ and $X \subset B \therefore x \in B$]

Thus, $x \in A \Rightarrow x \in B$

$\therefore A \subseteq B$ ...(1)

Now, let y be an arbitrary element of B. Then, there exists a subset, say Y, of set B such that $y \in Y$.

Now, $y \subset B \Rightarrow Y \in P(B)$

$\Rightarrow Y \in P(A)$ [$\because$ P(A) = P(B)]

$\Rightarrow Y \subset A$

$\Rightarrow Y \in A$

Thus, $y \in B \Rightarrow y \in A$

$\therefore B \subseteq A$

...(2)

From (1) and (2), we obtain A = B .

28. Prove that cos 20° cos 40° cos 60° cos 80° = $\frac{1}{16}$

L.H.S = cos 20° cos 40° cos 60° cos 80°

= $\frac{1}{2}\cos 20^\circ \cos 40^\circ \cos 80^\circ$ $\left[\because \cos 60^\circ = \frac{1}{2}\right]$

= $\frac{1}{4}(2\cos 20^\circ \cos 40^\circ)\cos 80^\circ$

= $\frac{1}{4}[\cos(20^\circ + 40^\circ)\cos(20^\circ - 40^\circ)]\cos 80^\circ$

[$\because$ 2 cos A cos B = cos (A + B) + cos (A – B)]

= $\frac{1}{4}(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ$ [$\because$ cos (–20°) = cos 20°]

= $\frac{1}{4}\ \frac{1}{2}\cos 80 \quad \cos 20 \ \ \cos 80$

= $\frac{1}{8}(\cos 80^\circ + 2\cos 20^\circ \cos 80^\circ)$

= $\frac{1}{8}[\cos 80^\circ + \cos(20^\circ + 80^\circ) + \cos(20^\circ - 80^\circ)]$

[$\because$ 2 cos A cos B = cos (A + B) + cos (A – B)]

= $\frac{1}{8}[\cos 80° + \cos 100° + \cos(-60°)] = \frac{1}{8}\left[\cos 80° - \cos 80° + \frac{1}{2}\right]$

= $\frac{1}{8} \quad \frac{1}{2} \quad \frac{1}{16}$ R.H.S

$\left[\because \cos 100° = \cos(180° - 80°) = -\cos 80° \text{ and } \cos(-60°) = \cos 60° = \frac{1}{2}\right]$

29. By the principle of mathematical induction, prove that $(1 + x)^n \geq 1 + nx$ for all $n \in N$ and $x > -1$.

Solution. Let P(n): $(1 + x)^n \geq 1 + nx$, for $x > -1$, $n \in N$ be the given statement. For n = 1, P(1): $(1 + x)^1 \geq 1 + x$, which is true, P(1) is true. Assume that P(k) $(1 + x)^k \geq 1 + kx$ holds. We shall prove that

P(k + 1): $(1 + x)^{k+1} \geq 1 + (k + 1)x$

Since $x > -1 \Rightarrow 1 + x > 0$

Multiplying both sides of (1) by 1 + x, we get

$(1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1 + kx + x + kx^2 \geq 1 + (k + 1)x$

[$\because k \in N, x^2 \geq 0 \Rightarrow kx^2 \geq 0$ for all $x \in R$]

$\therefore (1 + x)^{k+1} \geq 1 + (k + 1)x \Rightarrow$ P(k + 1) is also true. Hence by mathematical induction, P(n) holds for all $n \in N$.